Transformation of Systems of Linear Differential Equations.

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§1. Staeckel has shown* that the most general transformation, which converts a general homogeneous linear differential equation of order $m > 1$, into another of the same form and order, is

$$T: \quad x = f(\xi), \quad y = \phi(\xi)\eta,$$

where $f(\xi)$ and $\phi(\xi)$ are arbitrary functions of ξ. If $m = 1$ the most general transformation is

$$x = f(\xi), \quad y = \phi(\xi)\eta^{\lambda},$$

where λ is a constant.

In this paper we consider, more generally, a *system* of linear differential equations, and find the most general transformation which converts such a system into a system of the same order. The transformation thus found, of course, contains T as a special case. The method of investigation is essentially the same as that of Staeckel.

A theory of invariants of systems of linear differential equations, based on this general transformation, is now being worked out by the writer.

§2. Any system of n independent homogeneous linear differential equations, containing n unknown functions $y_1, y_2, \ldots\ldots y_n$ of x, and their derivatives of

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* Crelle's Journal für Mathematik, Bd. 111.

order 1, 2, ... m, can be written in the form

$$\left.\begin{array}{l}
y_i^{(m)} + \sum_{k=1}^{n} (p_{m-1,i,k} y_k^{(m-1)} + \ldots + p_{1ik} y_k' + p_{0ik} y_k) = 0, \\
\qquad (i = 1, 2. \ldots \lambda_1), \\
y_j^{(m-1)} + \sum_{k=1}^{n} (p_{m-2,j,k} y_k^{(m-2)} + \ldots + p_{1jk} y_k' + p_{0jk} y_k) = 0, \\
\qquad j = (\lambda_1 + 1, \lambda_1 + 2, \ldots \lambda_1 + \lambda_2), \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
y_\sigma' + \sum_{k=1}^{n} p_{0\sigma k} y_k = 0, \quad (\sigma = \lambda_{m-} + 1, \lambda_{m-1} + 2, \ldots \lambda_m), \\
\sum_{k=1}^{n} p_{0\tau k} y_k = 0, \quad (\tau = \lambda_m + 1, \lambda_m + 2, \ldots \lambda), \\
\qquad \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_m + \lambda = n.
\end{array}\right\} \tag{1}$$

The integration of such a system involves

$$\lambda_1 m + \lambda_2 (m-1) + \ldots + \lambda_m$$

arbitrary constants, and as such we can take the values of

$$\left.\begin{array}{l}
y_1 \quad \ldots y_{\lambda_1}, \; y_1', \ldots y_{\lambda_1}', \ldots y_1^{(m-1)}, \ldots y_{\lambda_1}^{(m-1)}, \\
y_{\lambda_1+1}, \ldots y_{\lambda_1+\lambda_2} \quad \ldots\ldots\ldots, \; y_{\lambda_1+1}^{(m-2)}, \ldots y_{\lambda_1+\lambda_2}^{(m-2)}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
y_{\lambda_1+\lambda_2+\ldots+\lambda_{m-1}+1}, \ldots\ldots\ldots\ldots \; y_{\lambda_1+\lambda_2+\ldots+\lambda_{m-1}+\lambda_m}
\end{array}\right\} \tag{I}$$

for an arbitrary value x_0 of x.

§3. We wish to find the most general transformation

$$y_i = g_i(\xi; \eta_1, \eta_2, \ldots \eta_n), \quad x = f(\xi; \eta_1, \eta_2, \ldots \eta_n), \tag{2}$$

which will transform (1) into a system of the same form (1′), which we can imagine written down, if in (1) we substitute Greek letters η, π and ξ for y, p and x.

Now from (2) we obtain

$$
\left.
\begin{aligned}
\frac{dy_i}{dx} &= \frac{\dfrac{\partial g_i}{\partial \xi} + \sum\limits_{\lambda=1}^{n} \dfrac{\partial g_i}{\partial \eta_\lambda}\dfrac{d\eta_\lambda}{d\xi}}{\dfrac{\partial f}{\partial \xi} + \sum\limits_{\lambda=1}^{n} \dfrac{\partial f}{\partial \eta_\lambda}\dfrac{d\eta_\lambda}{d\xi}} = \frac{Y_{i1}}{\sigma}, \\
\frac{d^2y_i}{dx^2} &= \frac{\sigma \dfrac{dY_{i1}}{d\xi} - Y_{i1}\dfrac{d\sigma}{d\xi}}{\sigma^3} = \frac{Y_{i2}}{\sigma^3}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{d^\mu y_i}{dx^\mu} &= \frac{Y_{i\mu}}{\sigma^{2\mu-1}},
\end{aligned}
\right\} \tag{3}
$$

where

$$\sigma = \frac{\partial f}{\partial \xi} + \sum_{\lambda=1}^{n} \frac{\partial f}{\partial \eta_\lambda}\frac{d\eta_\lambda}{d\xi}, \tag{4}$$

and Y_{i1}, Y_{i2}, etc., are defined by these equations, and are rational integral functions of $\eta'_\lambda, \eta''_\lambda, \ldots . \eta_\lambda^{(\mu)}$. In particular,

$$Y_{i2} = \sigma \frac{dY_{i1}}{d\xi} - Y_{i1}\frac{d\sigma}{d\xi},$$

and if we denote by $H_{i2\lambda}$ the coefficient of η''_λ in Y_{i2},

$$H_{i2\lambda} = \frac{\partial g_i}{\partial \eta_\lambda}\frac{\partial f}{\partial \xi} - \frac{\partial f}{\partial \eta_\lambda}\frac{\partial g_i}{\partial \xi} + \sum_{\mu=1}^{n}\left(\frac{\partial g_i}{\partial \eta_\lambda}\frac{\partial f}{\partial \eta_\mu} - \frac{\partial f}{\partial \eta_\lambda}\frac{\partial g_i}{\partial \eta_\mu}\right)\eta'_\mu. \tag{5}$$

$$(\lambda = 1, 2, \ldots . n).$$

These cannot all vanish identically, for else the functions f and g_i would not be dependent.

If we differentiate

$$\frac{d^{\mu-1}y}{dx^{\mu-1}} = \frac{Y_{i,\mu-1}}{\sigma^{2\mu-3}},$$

with respect to x, we get

$$\frac{d^\mu y}{dx^\mu} = \frac{\sigma \dfrac{dY_{i,\mu-1}}{d\xi} - (2\mu-3)\dfrac{d\sigma}{d\xi}Y_{i,\mu-1}}{\sigma^{2\mu-1}},$$

so that

$$Y_{i\mu} = \sigma \frac{dY_{i,\mu-}}{d\xi} - (2\mu-3)\frac{d\sigma}{d\xi}Y_{i,\mu-1}. \tag{6}$$

Denoting generally by $H_{i\mu\lambda}$ the coefficient of $\eta_\lambda^{(\mu)}$ in $Y_{i\mu}$, we have, therefore,

$$H_{i\mu\lambda} = \sigma H_{i,\,\mu-1,\,\lambda},$$

and hence

$$H_{i\mu\lambda} = \sigma^{\mu-2} H_{i2\lambda}, \qquad (\mu = 2, 3, \ldots\ldots m), \tag{7}$$

so that $H_{i\mu\lambda}$ is different from zero, if $H_{i2\lambda}$ is.

§4. If we substitute the values (2) and (3) in (1), we get

$$\left.\begin{aligned} Y_{im} + \sum_{k=1}^{n} (p_{m-1,\,i,\,k} Y_{k,\,m-1}\sigma^2 + p_{m-2,\,i,\,k} Y_{k,\,m-2}\sigma^4 + \ldots\ldots \\ + p_{1ik} Y_{k1}\sigma^{2m-2} + p_{0ik}\, g_k\, \sigma^{2m-1}) = 0, \\ \text{etc. } \ldots\ldots \text{ etc.} \qquad\qquad\qquad\qquad \end{aligned}\right\} \tag{8}$$

Now Y_{im} is linear in $\eta_1^{(m)}, \ldots\ldots \eta_n^{(m)}$, and actually contains at least one of the m^{th} derivatives, since at least one of the coefficients $H_{i2\lambda}$ and, therefore, at least one $H_{im\lambda}$ is different from zero.

Now, it must be possible to solve (8) for λ_1 derivatives of order m, λ_2 of order $m-1$, etc. For the transformed system is to be of the same order as (1). If the notation is so chosen, we shall, therefore, be able to express $\eta_1^{(m)}, \ldots\ldots \eta_{\lambda_1}^{(m)}$ in terms of lower derivatives, etc. Moreover, these expressions must be linear and homogeneous in $\eta_1, \ldots\ldots \eta_n$ and their derivatives, so that the transformed system may be of the same form as (1). If, then, in (8) itself, we divide each equation by the coefficient of one of the derivatives of highest order which occurs in it, the resulting equations must themselves be homogeneous and linear in $\eta_1, \ldots\ldots \eta_n$, etc.

Let $\eta_\lambda^{(m)}$ be a derivative which occurs in the i^{th} equation (8). Divide by $H_{im\lambda}$. Since the coefficients p_{abc} are arbitrary, each of the terms

$$\frac{Y_{k,\,m-1}\, p_{m-1,\,i,\,k}\, \sigma^2}{H_{m\lambda}}, \ldots\ldots \frac{Y_{k1}\, p_{1ik}\, \sigma^{2m-2}}{H_{im\lambda}}, \quad \frac{p_{0ik}\, g_k\, \sigma^{2m-1}}{H_{im\lambda}} \tag{9}$$

must be homogeneous and linear in $\eta_1, \ldots\ldots \eta_n, \eta_1', \ldots\ldots \eta_n'$, etc.

Now the last of these expressions is

$$\frac{p_{0ik}\,[f(\xi\,;\,\eta_1, \ldots\ldots \eta_n)]\; g_k\,[(\xi\,;\,\eta_1, \ldots\ldots \eta_n)]\left(\dfrac{\partial f}{\partial \xi} + \displaystyle\sum_{\lambda=1}^{n} \dfrac{\partial f}{\partial \eta_\lambda}\,\dfrac{d\eta_\lambda}{d\xi}\right)^{m+1}}{\dfrac{\partial g_i}{\partial \eta_\lambda}\,\dfrac{\partial f}{\partial \xi} - \dfrac{\partial f}{\partial \eta_\lambda}\,\dfrac{\partial g_i}{\partial \xi} + \displaystyle\sum_{\mu=1}^{n}\left(\dfrac{\partial g_i}{\partial \eta_\lambda}\,\dfrac{\partial f}{\partial \eta_\mu} - \dfrac{\partial g_i}{\partial \eta_\mu}\,\dfrac{\partial f}{\partial \eta_\lambda}\right)\dfrac{d\eta_\mu}{d\xi}}$$

This is linear and homogeneous in $\frac{d\eta_\lambda}{d\xi}$, when $m > 1$, only if

$$\frac{\partial f}{\partial \eta_\lambda} = 0, \qquad (\lambda = 1, 2, \ldots . n),$$

i. e., if x is a function of ξ only, say

$$x = f(\xi).$$

In that case, the expression

$$\frac{p_{0ik}[f(\xi)]\, g_k(\xi;\, \eta_1, \ldots . \eta_n) f'(\xi)^m}{\frac{\partial g_i}{\partial \eta_\lambda}}$$

must be a homogeneous linear function of $\eta_1, \ldots . \eta_n$. Moreover, the notation can always be so chosen that $\lambda = i$, so that $\eta_i^{(m)}$ occurs in the i^{th} equation (8). Then

$$\frac{p_{0ik}[f(\xi)]\, g_k(\xi;\, \eta_1, \ldots . \eta_n) f'(\xi)^m}{\frac{\partial g_i}{\partial \eta_i}} \qquad \begin{pmatrix} i = 1, 2, \ldots . \lambda_1 \\ k = 1, 2, \ldots . n \end{pmatrix} \tag{10}$$

must be linear and homogeneous in $\eta_1, \ldots . \eta_n$.

A similar examination of the $m - 2^{\text{nd}}$ expression (9),

$$\frac{Y_{k2}^{\cdot} p_{2ik} \sigma^{2m-4}}{H_{imi}},$$

shows that this expression is linear in $\eta_1', \ldots . \eta_n'$ if, and only if,

$$\frac{\partial^2 g_k}{\partial \eta_\lambda \partial \eta_\mu} = 0, \qquad (\lambda, \mu = 1, 2, \ldots . n),$$

so that

$$g_k = \alpha_{k1}(\xi)\eta_1 + \alpha_{k2}(\xi)\eta_2 + \ldots . + \alpha_{kn}(\xi)\eta_n + \alpha_{k0}(\xi).$$

But since (10) must be linear and homogeneous in $\eta_1, \ldots . \eta_n$, α_{k0} must vanish.

In the general case $m > 1$, we have, therefore, shown that the most general transformation which converts (1) into a system of the same order and degree is

$$\left.\begin{array}{c} x = f(\xi), \quad y_k = \alpha_{k1}(\xi)\eta_1 + \alpha_{k2}(\xi)\eta_2 + \ldots . + \alpha_{kn}(\xi)\eta_n, \\ (k = 1, 2, \ldots . n), \end{array}\right\} \tag{11}$$

and moreover, it is clear that every transformation of this form actually effects the required change, provided that

$$|\alpha_{ki}(\xi)| \neq 0.$$

§5. We still have to examine the particular case $m = 1$. But in this case it is better to adopt an entirely different method of proof.

Let
$$\frac{dy_k}{dx} = p_{k1} y_1 + \dots + p_{kn} y_n \qquad (k = 1, 2, \dots n) \quad (12)$$

be the given system. Let

$$y_{k1}, y_{k2}, \dots y_{kn} \qquad (k = 1, 2, \dots n) \quad (13)$$

be a simultaneous fundamental system of (12), so that the general solutions will be

$$y_k = c_1 y_{k1} + c_2 y_{k2} + \dots + c_n y_{kn}, \qquad (k = 1, 2, \dots n).$$

Any fundamental system of (12) is then of the form

$$\overline{y_{ki}} = c_{i1} y_{k1} + c_{i2} y_{k2} + \dots + c_{in} y_{kn}, \quad (i, k = 1, 2, \dots n), \quad (14)$$

where
$$\Delta = |c_{ij}| \neq 0, \qquad (i, j = 1, 2, \dots n),$$

and c_{ij} are arbitrary constants.

But we can write (12) in a different way. If we differentiate each equation (12) $n - 1$ times, we get

$$\frac{d^\lambda y_k}{dx^\lambda} = p_{k\lambda 1} y_1 + \dots + p_{k\lambda n} y_n, \quad (\lambda = 1, 2, \dots n). \quad (15)$$

By eliminating the $n - 1$ quantities y_i, $i \neq k$ from (15), we obtain n equations

$$r_{kn} \frac{d^n y_k}{dx^k} + r_{k,n-1} \frac{d^{n-1} y_k}{dx^{n-1}} + \dots + r_{k0} y_k = 0 \quad (16)$$
$$(k = 1, 2, \dots n)$$

for $y_1, \dots y_n$. In special cases these equations may be of lower than the n^{th} order, but in general they are not.

But there are relations between the equations (16), so that the general solution of one of them, say y_1, being known, the others are at once obtained in the form

$$y_k = s_{k0} y_1 + s_{k1} \frac{dy_1}{dx} + \dots + s_{k,n-1} \frac{d^{n-1} y_1}{dx^{n-1}}, \quad (17)$$
$$(k = 2, 3, \dots n).$$

This follows simply from (15) by putting $k=1$ and solving for $y_2, \ldots . y_n$. In other words, *equations* (16) *are cogredient*, as might also have been proved from the fact that the simultaneous fundamental systems of (12) are cogredient.

The general system (12) is, therefore, equivalent to the system

$$\left.\begin{aligned} & r_n \frac{d^n y_1}{dx_n} + r_{n-1} \frac{d^{n-1} y_1}{dx^{n-1}} + \ldots . + r_0 y_1 = 0, \\ & y_k = s_{k0} y_1 + s_{k1} \frac{dy_1}{dx} + \ldots . + s_{k,\, n-1} \frac{d^{n-1} y_1}{dx^{n-1}}, \quad (k = 2, 3, \ldots . n). \end{aligned}\right\} \tag{18}$$

If the transformation

$$y_k = g_k(\xi\,; \eta_1, \ldots . \eta_n), \quad x = f(\xi\,; \eta_1, \ldots . \eta_n) \tag{19}$$

converts (12) into a system of the same form and order

$$\frac{d\eta_k}{d\xi} = \pi_{k1} \eta_1 + \ldots . + \pi_{kn} \eta_n, \quad (k = 1, 2, \ldots . n), \tag{12a}$$

it must also convert (18) into a system of the same form and order

$$\left.\begin{aligned} & \rho_n \frac{d^n \eta_1}{d\xi^n} + \rho_{n-1} \frac{d^{n-1} \eta_1}{d\xi^{n-1}} + \ldots . + \rho_0 \eta_1 = 0, \\ & \eta_k = \sigma_{k0} \eta_1 + \sigma_{k1} \frac{d\eta_1}{d\xi} + \ldots . + \sigma_{k,\, n-1} \frac{d^{n-1} \eta_1}{d\xi^{n-1}}, \end{aligned}\right\} \tag{18a}$$

(18a) being equivalent to (12a) just as (18) is to (12).

But, by the method of the general case, it is now seen that the only transformations which convert (18) again into a linear system are, in general,

$$y_k = \alpha_{k1}(\xi)\, \eta_1 + \ldots . + \alpha_{kn}(\xi)\, \eta_n, \quad x = f(\xi),$$

provided that $n > 1$, and every such transformation does change (12) into a system of the same form.

For $n = m = 1$ the theorem is not true. Staeckel has settled this case. For particular cases, of course, there may be other transformations which effect the required transformation. For instance, if $m = 1$, $n > 1$, $p_{ik} = 0$ for $i \neq k$ and $p_{kk} \neq 0$. transformations of the form

$$x = f(\xi), \quad y_k = \phi_k(\xi)\, \eta_k^{\lambda_k},$$

where λ_k is a constant, are also permissible.

§6. We have the general theorem. *All transformations which convert a general system of n homogeneous linear differential equations into another of the same form and order, have the form*

$$x = f(\xi), \quad y_k = \alpha_{k1}(\xi)\eta_1 + \ldots + \alpha_{kn}(\xi)\eta_n, \quad (k = 1, 2, \ldots n),$$

where f, $\alpha_{k1} \ldots \alpha_{kn}$ are arbitrary functions of ξ. Only if $n = 1$, and if the single differential equation to which the system then reduces is of the first order, is there an exception, the most general transformation being in that case

$$x = f(\xi), \quad y = \alpha(\xi)\eta^{\lambda},$$

where λ is a constant.

This theorem can be extended to systems of non-linear homogeneous differential equations by a method analogous to the method employed for this purpose by Staeckel in the case of a single differential equation.

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